— Exercises —

1. **Convolution integrals.** Let x be a positive number and T be the triangle described by $\{(u,t) \in \mathbb{R}^2 | 0 \leq u \leq x, 0 \leq t \leq u\}$. Let f be a continuous function defined on $[0,x]$.

   (a) The convolution integral of f is the integral $\widehat{f}(x) = \int_T f$ (where f is seen as a function on T which does not depend on u). Show that

   $$\widehat{f}(x) = \int_0^x f(t)(x-t)dt$$

   (b) Compute $\widehat{f}(x)$ when $f(t) = e^{-t}$.

2. **Distance between a point and a curve.** Let f a $\mathcal{C}^1$ function on $\mathbb{R}^2$. We assume that $S = f^{-1}(\{0\})$ is non empty and the gradient of f vanishes nowhere on S. Recall that the distance $d(X,S)$ between the point $X = (x_0, y_0)$ and S is the infimum of $\{||X - s|| \, | s \in S\}$.

   (a) We assume that $(x_0, y_0) \notin S$. Recall that, since S is closed, $d(X,S) = d(X,(x,y))$ for some $(x,y) \in S$. Show that the vector $(x - x_0, y - y_0)$ is colinear to the gradient of f at (x,y).

   (b) Application: compute the distance between the curve $xy = 3$ and the origin of $\mathbb{R}^2$.

— Problems —

3. **Constrained maximum.** Let $n \geq 1$. Let $U_n = (\mathbb{R}_{\geq}0)^n$ and f_n the function defined on U_n by $f_n(x_1, \ldots, x_n) = -\sum_{j=1}^n x_j ln(x_j)$ with the convention that $yln(y) = 0$ when $y = 0$. We want to compute the maximum of f_n on U_n with the constraint $x_1 + \cdots + x_n = 1$.

   (a) Explain briefly why f_n is continuous on U_n for every $n \geq 1$.

   (b) Show that f_n admits a (global) maximum on $K_n = U_n \cap \{x | x_1 \cdots + x_n = 1\}$ for every $n \geq 1$.

   (c) Show by induction that the maximum of f_n on K_n is obtained at $\gamma_n = (\frac{1}{n}, \ldots, \frac{1}{n})$ and that $f_n(\gamma_n) = ln(n)$. *Hint: use the induction hypothesis to prove that the maximum of f_n is not obtained on the "boundary" $\partial K_n := \{ x \in K_n \mid \exists j, \ x_j = 0 \}$.*

4. **Lemoine point of a triangle.** Let T be a triangle whose edges have positive lengths a, b and c, and with area S. If M is a point of T, we call x (resp. y, z) the distance of M to edge of length a (resp. b,c). We want to find the Lemoine point of T, i.e. the point minimizing the sum $x^2 + y^2 + z^2$ of squared distances to the edges of T.

   (a) Show that we have $2S = ax + by + cz$.

   (b) Prove that the Lemoine point exists.

   (c) We assume that Lemoine point is in the interior of T. Compute its "coordinates" (x,y,z) using Lagrange method.

   (d) Show that Lemoine point is in the interior of T.

5. **(Optional) Volume of the n-ball.** We recall that the volume of a subset A of $\mathbb{R}^n$ is the value $\nu(A)$ of $\int_{\mathbb{R}^n} \chi_A$ (when this number exists). In what follows, $D(R,n)$ is the ball of radius R and centered at 0 of $\mathbb{R}^n$.

   (a) Compute $\nu(D(R,1))$ and $\nu(D(R,2))$.

   (b) We assume that $\nu(D(R,n)) = C_n R^n$ where C_n is a positive number which does not depend on R. Let $x = (x_1, \ldots, x_{n+2})$ be a point of $D(R, n+2)$. Using polar coordinates on x_{n+1} and x_{n+2}, show that $\nu(D(R,n+2)) = \int_0^R \int_0^{2\pi} r\nu(D(\sqrt{R^2 - r^2}, n))d\theta dr$.

   (c) Show that $\nu(D(R,n+2)) = \frac{2\pi R^2}{n}\nu(D(R,n))$